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Nonlinear Coupling of Waves in a Plasma in a Strong Dissipation Limit

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We study the nonlinear resonant coupling of two waves in a plasma for strong dissipation. We show that the corresponding system of differential equations has a saddle-focus fixed point and study its stable and unstable manifolds. The results we obtain suggest that the stochasticity which is numerically observed might be due to the existence of a spiral-type strange attractor.

KEY WORDS: Plasmas; drift wave turbulence; Shilnikov theorem; spiral-type attractors.

1. INTRODUCTION

1.1. Motivation

Plasmas are nonequilibrium media. Collective modes originating in the Coulombian interaction may develop and lead to drastic changes in the state of the plasma. However, nonlinear couplings of such waves with damped ones may saturate the instability at a level enabling the plasma to survive in a state which differs little from the initial one. Depending on the physical situation the physicists are faced with either of the following mode couplings:

(i) A large number of waves play a part in the system's evolution and interact. Statistical methods are then required to deal with the problem.

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(ii) Only a few waves rule the plasma's evolution. Such a situation usually occurs in small systems where boundary conditions lead to strong mode selection.

In the latter case, the system's evolution is given by a system of ordinary differential equations on waves amplitudes if we may neglect any nonlinear wave–particle interaction. Such an assumption is justified when the phase velocities are much larger than the thermal speed of electrons or when collisional processes are efficient enough to impede the evolution of the distribution functions of particles.

In this paper, we consider such a situation and study by pertubative methods in a strong dissipation limit the nonlinear resonant coupling of two waves with almost harmonic frequencies ω and $2\omega+\delta$. we shall assume that the high-frequency wave is damped and the low one is unstable, since in the opposite case the system's asymptotic state is just a lineary stable fixed point.

Such a model pertains to drift-wave turbulence which is of fundamental interest in the problem of fluctuations and anomalous transport in plasmas. The strong dissipation hypothesis is a relevant one as shown by Terry and Horton[1] and Vyshkind and Rabinovich.[2]

1.2. The Model

The high-frequency wave has complex amplitude A_u and linear growth rate $\gamma_u > 0$, the low-frequency one has amplitude A_s and linear damping rate $\gamma_s < 0$. If the amplitudes remain small and the evolution is slow, we may describe the coupling by the following set of equations which originates in a linearization of the dispersion relations[3]:

$$i\left(\frac{dA_u}{dt} - \gamma_u A_u\right) = VA_s^2 \exp(i\delta t)$$

$$i\left(\frac{dA_s}{dt} - \gamma_s A_s\right) = VA_u A_s^* \exp(-i\delta t)$$

where $\delta = \omega_u - 2\omega_s$ and V is a coupling parameter. Since no resonant wave particle interaction is considered, V is a real number and we set $V = 1$. The star denotes complex conjugaison.

If we set

$$A_u = a_u \exp(i\Phi_u), \qquad a_u \in \mathscr{R}^+$$

$$A_s = a_s \exp(i\Phi_s), \qquad a_s \in \mathscr{R}^+$$

$$\phi = \Phi_u - 2\Phi_s - \delta t,$$

we obtain the following equations on real amplitudes and the relative phase ϕ:

$$\frac{da_u}{dt} = \gamma_u a_u - a_s^2 \sin \phi$$

$$\frac{da_s}{dt} = \gamma_s a_s + a_s a_u \sin \phi$$

$$\frac{d\phi}{dt} = -\delta + \frac{2a_u^2 - a_s^2}{a_u} \cos \phi$$

$\gamma_u + \gamma_s < 0$ is the condition for the system to be dissipative. In the present paper, we shall study by pertubative methods the strong dissipation limit $-\gamma_s \gg \gamma_u$. The energy-conserving case $\gamma_u = \gamma_s = 0$ could also lead to an interesting study (see Appendix 1).

To study the strong dissipation case limit we introduce the following:

(i) The dissipation parameter $p = (\gamma_s - \gamma_u)/\gamma_s$ Since $1 \leqslant p < 2$, we shall often set $p = 1 + \varepsilon$ and ε will be the perturbation parameter.
(ii) The frequency mismatch parameter $\mu = -\delta/p\gamma_s$
(iii) The rescaled time:

$$u = \int_0^t (a_u^2 + a_s^2)^{1/2}\, dt$$

(iv) The new variables:

$$x = \left(\frac{a_u^2}{a_u^2 + a_s^2}\right)^{1/2} \sin \phi$$

$$y = \left(\frac{a_u^2}{a_u^2 + a_s^2}\right)^{1/2} \cos \phi$$

$$z = \frac{-p\gamma_s}{(a_u^2 + a_s^2)^{1/2}}$$

Notice that z contains information on the total energy, x and y information on the relative phase and the distribution of energy between the two waves.

Thus we obtain the following system with polynomial right-hand member:

$$\frac{dx}{du} = (xz - 1)(1 - x^2 - y^2) + 2y\left(y - \frac{\mu z}{2}\right)$$

$$\frac{dy}{du} = yz(1 - x^2 - y^2) - 2x\left(y - \frac{\mu z}{2}\right)$$

$$\frac{dz}{du} = z^2\left(\frac{1}{p} - x^2 - y^2\right)$$

Notice that p appears only in the third equation (energy equation), and μ in the first and second ones (phase equations).

The natural phase space is the semi-infinite cylindrical volume $0 \leqslant x^2 + y^2 \leqslant 1$, $0 \leqslant z$.

In Section 2 we study the infinite dissipation limit $\varepsilon = 0$. Section 3 deals with the strong dissipation case $0 < \varepsilon \ll 1$. In Section 3.1 we prove the existence of an invariant compact set. In Section 3.2 we show that the system has a saddle focus fixed point. In Section 3.3 we study by pertubation its stable manifold. In Section 3.4 a similar study is performed on the unstable manifold.

Those results suggest that the fixed point could be homoclinic for certain parameter values and that the stochasticity numerically observed[4,5] might be due to a spiral type strange attractor.

The proofs require lengthy calculations. An outline of the proofs is given in the bulk of the paper. Some technical points are explained in more detail in the appendices.

2. THE INFINITE DISSIPATION LIMIT: $p = 1$

For $p = 1$ the system writes

$$\frac{dx}{du} = (xz - 1)(1 - x^2 - y^2) + 2y\left(y - \frac{\mu z}{2}\right)$$

$$\frac{dy}{du} = yz(1 - x^2 - y^2) - 2x\left(y - \frac{\mu z}{2}\right)$$

$$\frac{dz}{du} = z^2(1 - x^2 - y^2)$$

Since it is invariant under the transformation of μ and y into $-\mu$ and $-y$ we shall assume $\mu \geqslant 0$.

Three manifolds are left invariant by the flow: the semi-infinite cylinder $x^2 + y^2 = 1$, $z \geqslant 0$; the plane $y = \mu(z/2)$ and the infinite energy plane

$z=0$. These three manifolds enclose for $\mu>0$ an invariant compact set K (see Fig. 1). We shall now study the fixed points of the flow in this invariant volume (Section 2.1) and the restrictions of the flow on the invariant manifolds $z=0$ (Section 2.2) and $y=\mu(z/2)$ (Section 2) for $\mu>0$.

2.1. The Fixed Points

The invariant manifolds $x^2+y^2=1$ and $y=\mu(z/2)$ intersect along a half-ellipse E of degenerate fixed points. All of them are neutral along E. Let (x_c, y_c, z_c) be the coordinates of any such fixed point.

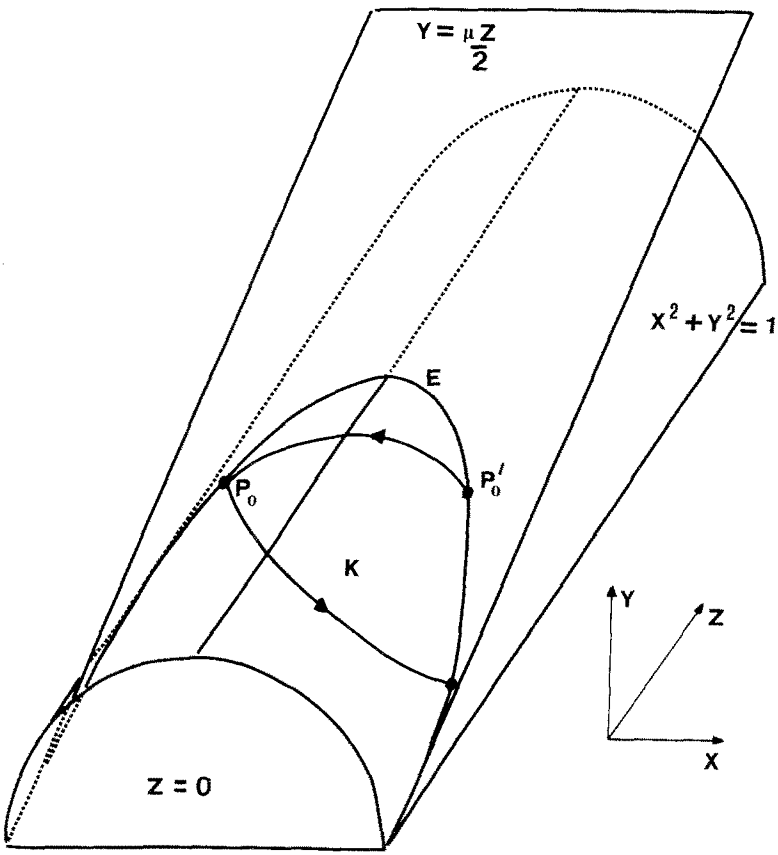

Fig. 1. Phase space of the system with the invariant manifolds for $\varepsilon=0$: $y=\mu(z/2)$, $x^2+y^2=1$, $z=0$ and the half-ellipse of fixed points E. One fixed point is depicted together with its stable and unstable manifolds.

(i) Any fixed point with $x_c > 0$ and $z_c < 1/2[1 + (\mu/4)^2]^{1/2}$ is linearly unstable (see Section 2.3 and Fig. 3 below).
   (a) Its stable manifold is the arc of circle $x^2 + y^2 = 1$, $z = z_c$, $y > 0$ connecting it with the fixed point $(-x_c, y_c, z_c)$ (see Fig. 1).
   (b) Its unstable manifold is a curve, lying in the plane $y = \mu(z/2)$ which connects it with a fixed point (x'_c, y'_c, z'_c), $z'_c > 1/2[1 + (\mu/4)^2]^{1/2}$ (see Fig. 1 and Fig. 3 below).

(ii) The fixed points with $x_c > 0$ and $z_c > 1/3[1 + (\mu/6)^2]^{1/2}$ have two-dimensional stable manifolds (see Fig. 3 below)

(iii) Any fixed point with $x_c < 0$ is unstable.
   (a) Its unstable manifold is the arc of circle $x^2 + y^2 = 1$, $z = z_c$, $y > 0$
   (b) Its stable manifold lies in the plane $y = \mu(z/2)$ (see Fig. 3 below).

An other fixed point of center type $(0, 1/\sqrt{3}, 0)$ lies in the invariant plane $z = 0$ (see Section 2.2 and Fig. 2 below).

2.2. The Flow on the Invariant Plane $z = 0$

For $z = 0$ the system writes

$$\frac{dx}{du} = -1 + x^2 + 3y^2$$

$$\frac{dy}{du} = -2xy$$

We study the flow in the region $y \geqslant 0$. All orbits are cycles except the half circle $x^2 + y^2 = 1$ and the line $y = 0$ which are invariant manifolds for the heteroclinic fixed points $(\pm 1, 0)$ (see Fig. 2). $(0, 1/\sqrt{3})$ is a center fixed point.

2.3. The Flow on the Invariant Surface $y = \mu(z/2)$

On $y = \mu(z/2)$ the system writes

$$\frac{dx}{du} = (xz - 1)(1 - x^2 - y^2)$$

$$\frac{dz}{du} = z^2(1 - x^2 - y^2)$$

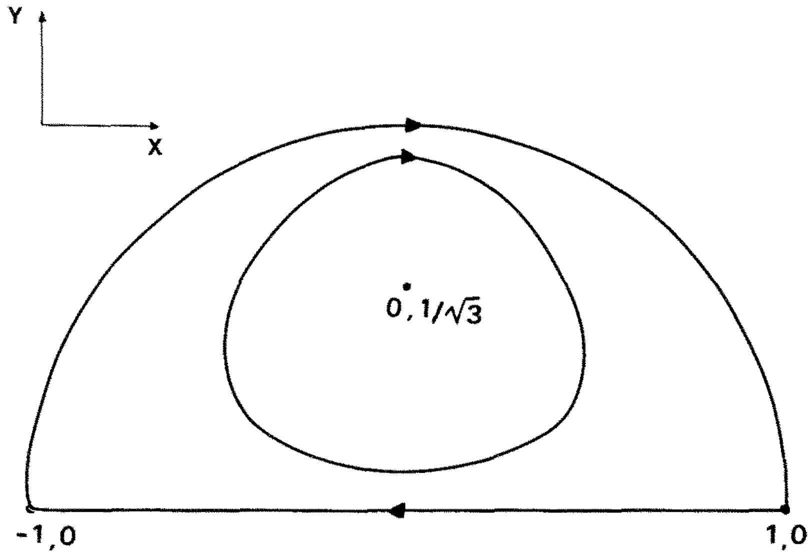

Fig. 2. Flow on the invariant manifold $z = 0$. The two heteroclinic fixed points and the center fixed points are depicted.

Therefore z increases on every trajectory. Those trajectories are stable or unstable manifolds of fixed points (see Section 2.2). Their equation $x(z)$ is

$$x(z) = \frac{1}{2z} + \frac{z}{z_0}\left[\left(1 - \frac{\mu^2 z_0^2}{4}\right)^{1/2} - \frac{1}{2z_0}\right]$$

where z_0 is the z coordinate of the point where the curve reaches the half ellipse E for $x > 0$. For $0 < \mu < 1$ some typical trajectories are depicted in Fig. 3.

3. THE STRONG DISSIPATION REGIME

We now study the system for $0 < \varepsilon \ll 1$. We show the existence of a compact set (Section 3.1) and a saddle-focus fixed point (Section 3.2). Then we study by pertubation the stable and unstable manifolds of this point.

3.1. The Compact Invariant Set

For $\varepsilon > 0$, $y = \mu(z/2)$ is no longer an invariant manifold. However, the vector field is transverse to this plane and points toward low values of z. Therefore, if $\{g_u, u \in \mathscr{R}\}$ denotes the flow, $K_\infty = \bigcap_{u > 0} g_u(K)$ is a nonempty

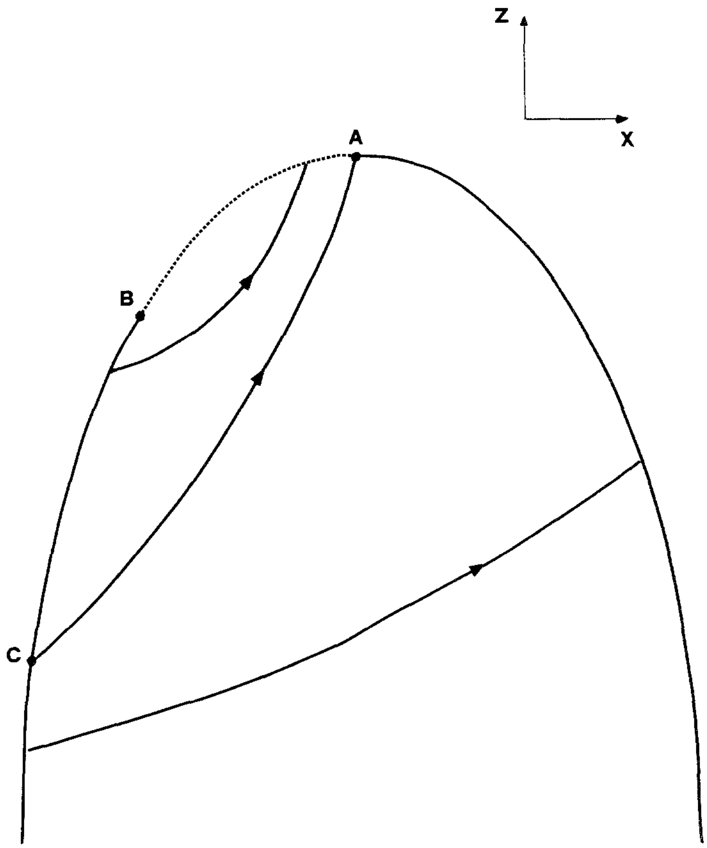

Fig. 3. Flow on the invariant manifold $y = \mu(z/2)$ for $\varepsilon = 0$. On the ellipse E, the continuous line denotes the fixed points which have dimension 1 stable manifolds, the dotted line those with dimension 2 stable manifolds. Some typical trajectories are shown together with some important points: A has coordinates $(0, 2/\mu)$; B, $\{(1/2)[(1+\mu)^{1/2} - (1-\mu)^{1/2}, (1/\mu)[(1+\mu)^{1/2} + (1-\mu)^{1/2}]\}$; C, $[1/(1+\mu^2/4)^{1/2}, 1/2(1+\mu^2/4)^{1/2}]$.

compact connected invariant set enclosed in the invariant set K we considered previously (see Section 2). We are interested in the dynamics of the flow in K_∞.

3.2. The Saddle Focus Fixed Point

For $\varepsilon > 0$ the flow has four fixed points. The three fixed points in the plane $z = 0$ previously defined and a fixed point P_ε located inside K_∞. Since $y = \mu(z/2)$ is no longer invariant the ellipse of degenerate fixed points has vanished.

P_ε has coordinates

$$\frac{1}{\alpha(1+\varepsilon)^{1/2}}, \qquad \frac{\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}, \qquad \frac{(1+\varepsilon)^{1/2}}{\alpha},$$

$$\text{where} \quad \alpha = \left[1 + \frac{\mu^2(1+\varepsilon^2)}{(2-\varepsilon)^2}\right]^{1/2}$$

For $\mu < \mu_0$ small enough, the eigenvalues of the linearized flow in the neighborhood of P_ε write for ε small enough:

$$\gamma = -2\alpha_0^2 + \frac{\varepsilon}{32}(\mu^4 - 8\mu^2 + 48) + O(\varepsilon^2)$$

$$\beta_\pm = \left[\frac{\varepsilon}{2}\alpha_0^2(2-\alpha_0^2) + O(\varepsilon^2)\right] \pm i\left[\sqrt{\varepsilon}\,\frac{\alpha_0}{2}(-\alpha_0^6 + 12\alpha_0^4 + 4\alpha_0^2 - 2)^{1/2} + O(\varepsilon)\right]$$

where

$$\alpha_0 = \left(1 + \frac{\mu^2}{4}\right)$$

Since $\gamma < 0$ and $\mathscr{R}(\beta_\pm) > 0$, P_ε is a saddle focus fixed point. Notice that $-\gamma > \mathscr{R}(\beta_\pm)$, a condition required for the use of Shilnikov theorem.[6,7]

Notice that for higher value of $\mu(\mu \sim 2)$, P_ε becomes stable through a Hopf bifurcation.

When ε goes to 0, P_ε moves to the boundary fixed point $P_0(1/\alpha_0, \mu/2\alpha_0, 1/\alpha_0)$ which belongs to the ellipse E. Therefore it is relevant to study the stable and unstable manifolds of P_ε by pertubation.

3.3. The Stable Manifold

For $\varepsilon = 0$ the fixed point P_0 has for stable manifold the arc of circle Γ_0' $(x^2 + y^2 = 1, z = 1/\alpha_0)$ which connects it with the fixed point $P_0'(-1/\alpha_0, \mu/2\alpha_0, 1/\alpha_0)$. This latter fixed point's stable manifold Γ_0'' lies in the plane $y = \mu(z/2)$ (see Fig. 4). Its equation writes

$$x(z) = \frac{1}{2z} - \left(1 + \frac{\alpha_0^2}{2}\right) z$$

For $\varepsilon > 0$ small enough, the upper branch Γ_ε of the stable manifold of P_ε remains close to the curve $\Gamma_0 = \Gamma_0' \cup \Gamma_0''$ in the domain $z > 1/(2 + \alpha_0^2)^{1/2}$ (see Fig. 4). To prove that result we proceed in four steps:

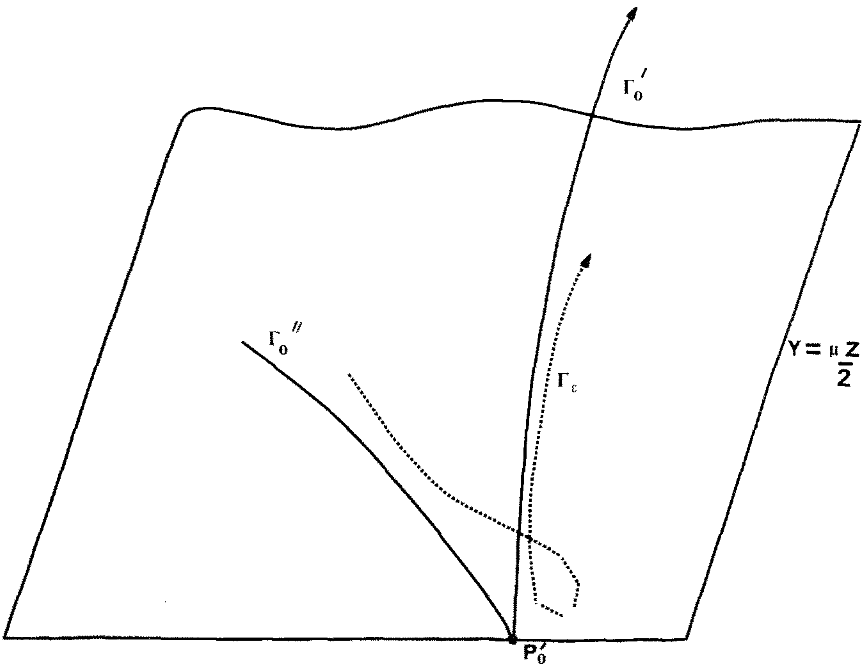

Fig. 4. Pertubation of the upper branch of the stable manifold of P_ε. The continuous curve is the unperturbed manifold Γ_0, the dotted line is the perturbed manifold Γ_ε.

(i) *Step One.* (a) By standard perturbation method we prove that for x remaining at a finite distance of $-1/\alpha_0$ we have a converging pertubation expansion for the equations $y(x)$, $z(x)$ of Γ_ε, for μ small enough and $\varepsilon < \varepsilon_0(\mu)$, which writes at first order in ε:

$$y = (1-x^2)^{1/2} + \frac{\varepsilon\mu^2 x^2}{4\alpha_0^2(1-x^2)^{1/2}} - \frac{\varepsilon}{\alpha_0}\left(\frac{1+x}{1-x}\right)^{1/2} \frac{1}{\alpha_0 + x - (\mu/2)(1-x^2)^{1/2}} + O(\varepsilon^2)$$

$$z = \frac{1}{\alpha_0} + \frac{\varepsilon}{2\alpha_0^3} - \frac{\varepsilon}{\alpha_0^3}\left\{\frac{\mu}{2} + \left[\frac{\alpha_0 - x + (\mu/2)(1+x^2)^{1/2}}{\alpha_0 + x - (\mu/2)(1-x^2)^{1/2}}\right]^2\right\}^2 + O(\varepsilon^2)$$

x is obviously the right geometric parameter of the curve until we reach the neighborhood of P_0' since the unperturbed trajectory is the arc of circle $y = (1-x^2)^{1/2}$.

(b) When x approches $-1/\alpha_0$ the pertubation expansion blows up owing to terms of the form $\phi(x)/[\alpha_0+x-(\mu/2)(1-x^2)^{1/2}]^n$ in the expansion. It comes from x being no longer a convenient geometric parameter (see Fig. 4). However we can prove that for $x>-1/\alpha_0(1-\varepsilon^{1/3})$ Γ_ε remains in a $\varepsilon^{1/3}$ neighbourhood of Γ_0.

(ii) *Steps Two and Three.* To proceed further and study Γ_ε in a neighborhood of P_0' we must adopt y as new geometric parameter of Γ_ε. We study Γ_ε in two steps owing to the singular behavior at P_0' where Γ_0 is not differentiable (see Fig. 5a and 5b).

(a) For

$$\frac{\mu}{2\alpha_0}<y<\frac{\mu}{2\alpha_0}\left[1+\frac{4\varepsilon^{1/3}}{\mu^2}\right]$$

Γ_0' is an arc of a circle, Γ_ε crosses the plane

$$y=\frac{\mu}{2\alpha_0}\left[1+\frac{4\varepsilon^{1|3}}{\mu^2}\right]$$

at a point

$$x=-\frac{1}{\alpha_0}(1-\beta\varepsilon^{1/3}) \qquad \text{where} \quad \beta=1+\frac{\mu^4}{16}+O\,[\varepsilon^{1/3}+\mu^6]$$

$$z=\frac{1}{\alpha_0}(1-\beta'\varepsilon^{1/3}) \qquad \text{where} \quad \beta'=\mu^2+O[\varepsilon^{1/3}+\mu^4]$$

It remains in a $\varepsilon^{1/3}$ neighborhood of Γ_0' for $y>\mu/2\alpha_0$ and crosses the plane $y=\mu/2\alpha_0$ at a point

$$x=-\frac{1}{\alpha_0}[1-\gamma\varepsilon^{1/3}] \qquad \text{where} \quad \gamma=2^{2/3}+O[\varepsilon^{1/3}+\mu^2]$$

$$z=\frac{1}{\alpha_0}[1-\gamma'\varepsilon^{1/3}] \qquad \text{where} \quad \gamma'=2^{1/3}+O[\varepsilon^{1/3}+\mu^2]$$

The proof is easy. It is given in Appendix 2 and relies on the behavior of the unperturbed flow near P_0'.

(b) For

$$\frac{\mu}{2\alpha_0}(1-\varepsilon^{1/3})<y<\frac{\mu}{2\alpha_0}$$

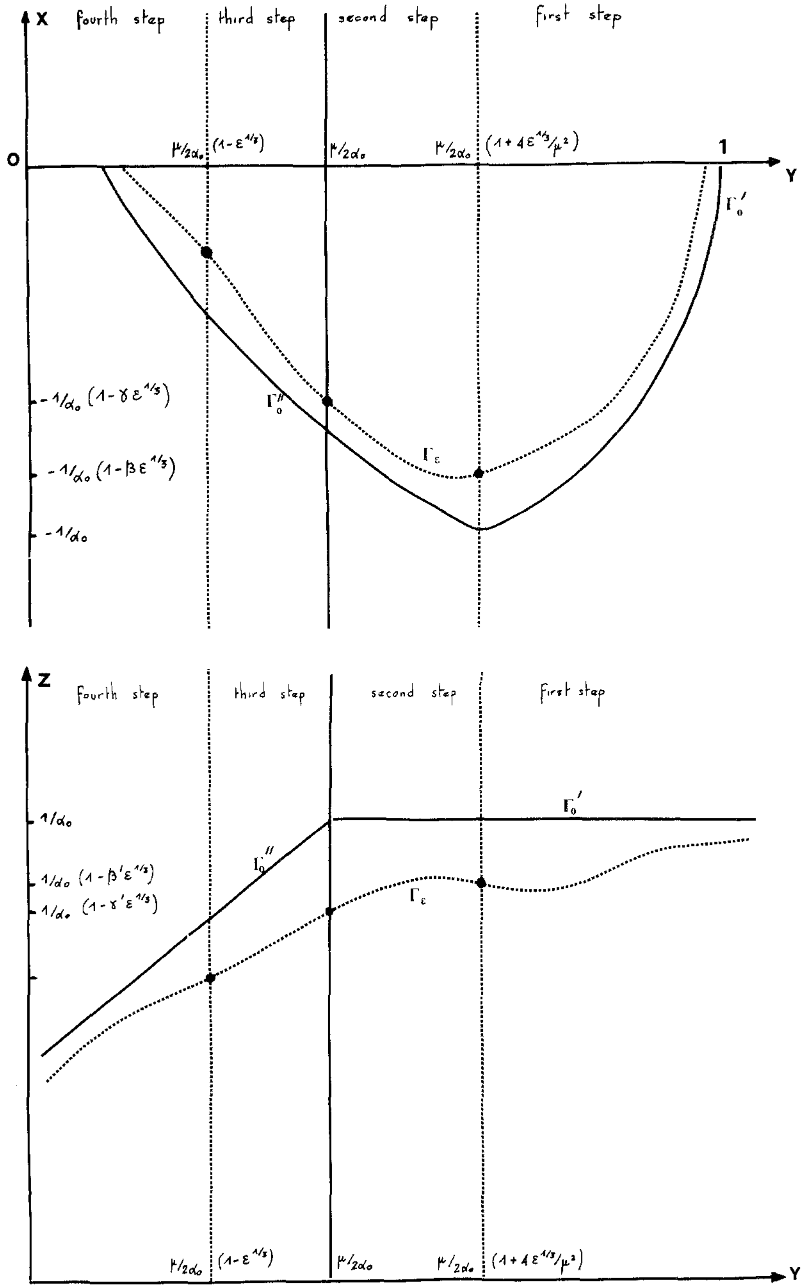

Fig. 5. Successive steps in the study of Γ_ε near P_0' depicted for variables x (Figure 5a) and z (Figure 5b).

We prove in a similar way that Γ_ε remains in an $\varepsilon^{1/3}$ neighborhood of Γ_0 the equations of which write

$$\tilde{x}(y) = \frac{\mu}{4y} - \left[1 + \frac{\alpha_0^2}{2}\right]\frac{2y}{\mu}$$

$$\tilde{z}(y) = \frac{2y}{\mu}$$

(iv) *Step Four.* In a final step we use standard perturbation methods to prove that Γ_ε remains in a $\varepsilon^{1/3}$ neighborhood of Γ_0'' until it crosses the plane $z = 1/(2+\alpha_0^2)^{1/2}$. (where $x(z) = 0$). In that last step we use z as the convenient geometric parameter (see Fig. 4).

3.4. The Unstable Manifold

For $\varepsilon = 0$, the plane $y = \mu(z/2)$ is the center unstable manifold of the fixed point P_0. We may study by perturbation the unstable manifold of P_ε and examine its behavior in a neighborhood of P_0'.

First of all, we show that the unstable manifold is an analytic manifold defined on the rectangle:

$$\frac{1}{\alpha(1+\varepsilon)^{1/2}} - \mu \leqslant x \leqslant \frac{1}{\alpha(1+\varepsilon)^{1/2}} + \mu\varepsilon$$

$$\frac{1}{3\left(1+\dfrac{\mu^2}{36}\right)^{1/2}} - \frac{1}{\left(1+\dfrac{\mu^2}{4}\right)^{1/2}} + \frac{(1+\varepsilon)^{1/2}}{\alpha} - \mu \leqslant z \leqslant \frac{(1+\varepsilon)^{1/2}}{\alpha} + \mu\varepsilon$$

For $\mu \leqslant \mu_0$ small enough and ε small enough, its equation writes

$$y - \mu\frac{z}{2} = \Psi_{\mu,\varepsilon}(x, z) \qquad \text{where} \quad 0 \leqslant \Psi_{\mu,\varepsilon} \leqslant \varepsilon\frac{\mu}{2}$$

(see Appendix 2)

To study the unstable manifold of P_ε in a neighborhood of P_0' we proceed as follows:

For $\varepsilon = 0$, the stable manifold of P_0' originates from the point $Q\ [1/(1+\mu^2/36)^{1/2},\ (\mu/6)/(1+\mu^2/36)^{1/2},\ (1/3)/(1+\mu^2/36)^{1/2}]$ (see Fig. 6).

We may study by pertubation for ε small enough the curves lying on the unstable manifold and originating in a neighborhood of Q until they reach a neighborhood of P_0'.

More precisely we show that the trajectory $x_a(z)$, $y_a(z)$ on the unstable

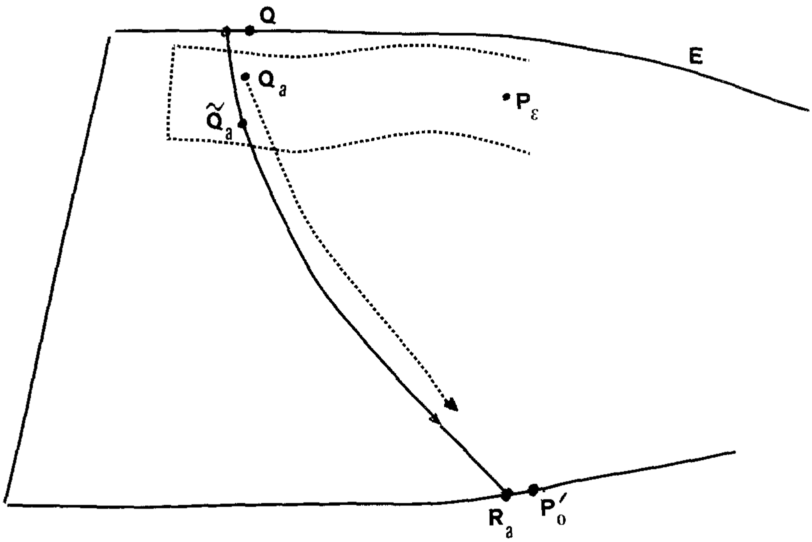

Fig. 6. Perturbative study of curves originating near Q and reaching E near P'_0. The dotted lines correspond to the unstable manifols of P_ε and the perturbed trajectory originating on that manifold in Q_a.

manifold of P_ε originating from the points $Q_a(1-\mu, \Psi_{\varepsilon,\mu}(1-\mu, \frac{1}{3}+a\mu) + (\mu/2)(\frac{1}{3}+a\mu), \frac{1}{3}+a\mu)$, $0 \leqslant a \leqslant 1$ remain close to the corresponding curves $\widetilde{x_a}(z), \widetilde{y_a}(z)$ on $y=\mu(z/2)$ originating from the point $Q_a(1-\mu, (\mu/2)(\frac{1}{3}+a\mu), \frac{1}{3}+a\mu)$ (see Fig. 6).

We have then for $\frac{1}{3}+a\mu \leqslant z \leqslant z_a - \varepsilon^{1/3}$:

$$|x_a(z) - \tilde{x}_a(z)| \leqslant N\varepsilon^{2/3}$$
$$|y_a(z) - \tilde{y}_a(z)| \leqslant N\varepsilon^{2/3} \qquad \text{for some positive number } N$$

where z_a is the z coordinates of the point R_a where $x_a(z)$, $y_a(z)$ reaches the ellipse E for negative values of x (see Fig. 6).

That result requires lengthy but straightforward calculations.

In a second step we study the perturbed trajectories after their crossing of the surface $z = z_a - \varepsilon^{1/3}$. For this purpose we use the coordinates $s = 1 - x^2 - y^2$, $t = y - \mu(z/2)$, z. The crossing of $z = z_a - \varepsilon^{1/3}$ occurs at some point $s = \delta\varepsilon^{1/3}$, $t = \delta'\varepsilon^{1/3}$ and we obtain for $t_e[\delta'\varepsilon^{2/3}, \eta]$ where η is some small positive number:

$$0 \leqslant s(t) \leqslant \theta\varepsilon^{1/3}, \qquad \theta \text{ some positive real number}$$

$$z_a - 3\varepsilon^{1/3} \leqslant z(t) \leqslant z_a \left\{1 + \varepsilon^{1/3}\log(\varepsilon)\left[\left(\frac{1}{z_a^2} - \frac{\mu^2}{4}\right)^{1/2} - \frac{1}{2z_a^2}\right]\right\}$$

Therefore any perturbed curve corresponding to the initial point Q_a remains in a $\varepsilon^{1/3}$ neighborhood of the unperturbed curve $x^2+y^2=1$, $z=z_a$ in the region near P'_0.

4. CONCLUSION

We have thus proved that for $\mu<\mu_0$ small enough and $\varepsilon<\varepsilon_0(\mu)$ the dimension one stable and dimension two unstable manifolds of the saddle focus fixed point A are defined and close together in a neighborhood of the point P'_0 (see Fig. 7). Therefore we may expect an homoclinic tangency to occur for certain values of the parameters. Then the stable manifold will be contained in the unstable one.

In such cases the assumptions of Shilnikov theorem[6,7] will be satisfied and in every neighborhood of the homoclinic orbit $\Gamma_{\mu,\varepsilon}$ there will exist a countable set of unstable periodic orbits of saddle type and a subsystem of trajectories in one-to-one correspondence with a shift with an infinite number of symbols. We know too that for nearby values of the parameters a finite number of these horseshoes do persist[7,8] although the homoclinic orbit has disappeared.

Such a situation might even give rise to a spiral type strange attractor[9] and we think that the chaos which has been numerically observed for high value of dissipation and small mismatch could possibly be explained in such terms.

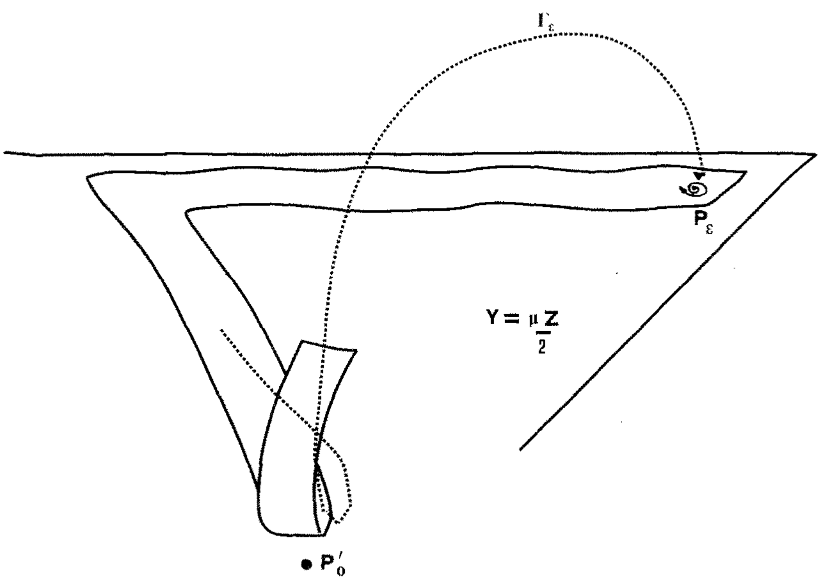

Fig. 7. Stable (dotted line) and unstable manifolds of P_ε for μ and ε small enough.

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APPENDIX 1: THE ENERGY-CONSERVING CASE

If we introduce the following set of variables:

$E = a_s^2 + a_u^2$	total energy
$B = \dfrac{a_u^2}{a_s^2 + a_u^2}$	which describes the distribution of energy between the two waves
ϕ	relative phase

together with the rescaled time

$$u = \int_0^t [M(t)]^{1/2}\, dt$$

we obtain

$$\frac{dE}{du} = 2\sqrt{E}\,[\gamma_u B + \gamma_s(1 - B)]$$

$$\frac{dB}{du} = -2\sqrt{B}\,(1 - B)\sin\phi + \frac{2B(1 - B)(\gamma_u - \gamma_s)}{\sqrt{E}}$$

$$\frac{d\phi}{du} = \frac{3B - 1}{\sqrt{B}}\cos\phi - \frac{\delta}{\sqrt{E}}$$

When $\gamma_s = \gamma_u = 0$ the energy is conserved and the system possesses on any constant energy surface ($E > 0$) a Hamilton function:

$$H = -2\sqrt{B}\,(1 - B)\cos\phi - \frac{\delta B}{\sqrt{E}}$$

with conjugate variables B and ϕ:

$$\frac{dB}{du} = -\frac{\partial H}{\partial \phi}$$

$$\frac{d\phi}{du} = \frac{\partial H}{\partial B}$$

The situation $0<\gamma_0\ll 1$, $0< -\gamma_s \ll 1$ may be studied by pertubation of the integrable limit case.

APPENDIX 2

We study Γ_ε for $\mu/2\alpha_0 \leqslant y \leqslant (\mu/2\alpha_0)(1+4\varepsilon^{1/3}/\mu^2)$ and we set $v=(y-\mu/2\alpha_0)\,\varepsilon^{-1/3}$. $\tilde{x}(v)=-(1-y^2)^{1/2}$, $\tilde{z}(v)=1/\alpha_0$ are the equations of Γ_0''.

The equations for Γ_ε write

$$x(v)=-(1-y^2)^{1/2}+\varepsilon^{1/3}X$$

$$z(v)=\frac{1}{\alpha_0}+\varepsilon^{1/3}Z$$

where X and Z satisfy for $v\in(0, 2/\mu\alpha_0)$

$$\frac{dX}{dv}=$$

$$\frac{X\left[(2\tilde{x}+\varepsilon^{1/3}X)(\tilde{z}-\tilde{x}+\varepsilon^{1/3}X\tilde{x}\tilde{z}+\varepsilon^{1/3}Z)+2\left(\frac{\mu}{2\alpha_0}+\varepsilon^{1/3}v\right)\left(\varepsilon^{1/3}v-\frac{\mu}{2}\varepsilon^{1/3}Z\right)\right]}{\tilde{x}X\left(\frac{\mu}{2\alpha_0}+\varepsilon^{1/3}v\right)(\tilde{z}+\varepsilon^{1/3}Z)(2\tilde{x}+\varepsilon^{1/3}X)+2\tilde{x}(\tilde{x}+\varepsilon^{1/3}X)\left(v-\frac{\mu}{2}Z\right)}$$

$$\frac{dZ}{dv}=\frac{(\tilde{z}+\varepsilon^{1/3}Z)^2\left(\frac{\varepsilon^{2/3}}{1+\varepsilon}+2\tilde{x}X+\varepsilon^{1/3}X^2\right)}{X\left(\frac{\mu}{2\alpha_0}+\varepsilon^{1/3}v\right)(\tilde{z}+\varepsilon^{1/3}Z)(2\tilde{x}+\varepsilon^{1/3}X)+2(\tilde{x}+\varepsilon^{1/3}X)\left(v-\frac{\mu}{2}Z\right)}$$

together with the initial conditions

$$X\left(\frac{2}{\mu\alpha_0}\right)=\tilde{\beta},\qquad Z\left(\frac{2}{\mu\alpha_0}\right)=-\frac{\beta'}{\alpha_0}$$

where $\tilde{\beta}=(\beta-1)/\alpha_0+O(\varepsilon^{1/3})$

We have to prove that the solutions exist for $v\in[0, 2/\mu\alpha_0]$. Actually it is sufficient to prove that for the lowest-order system

$$\frac{dX}{dv}=-\frac{4X}{\mu X/\alpha_0^2+2v-\mu Z}$$

$$\frac{dZ}{dv}=\frac{2X/\alpha_0^2}{\mu X/\alpha_0^2+2v-\mu Z}$$

a solution with the same initial conditions exists for $v\in[0, 2/\mu\alpha_0]$.

If we set

$$w = 2\mu\alpha_0 v$$
$$S = \frac{\mu^2}{\alpha_0} X$$
$$T = \mu^2 \alpha_0 Z$$

we have

$$\frac{dS}{dw} = -\frac{2S}{w+S-T}, \qquad \frac{dT}{dw} = \frac{S}{w+S-T}, \qquad w \in [0,4]$$

with initial conditions: $S(4) = \mu^2\tilde{\beta}/\alpha_0 > 0$, $T(4) = -\mu^2\beta' < 0$.

Since $d(S+2T)/dw = 0$, we have, setting $k = \mu^2[\tilde{\beta}/\alpha_0 - 2\beta']$, $S + 2T = k < 0$ on the trajectory and

$$\frac{dS}{dw} = \frac{-2S}{w + 3S/2 - k/2}$$

which has a positive decreasing solution on $[0,4]$ for the given initial condition.

Since $T = k/2 - S/2$, T also exists on $[0,4]$ for the initial condition $T(4) = -\mu^2\beta'$. T is a negative increasing function on $[0,4]$.

We may estimate $S(0)$ and $T(0)$: If we set

$$f = \sqrt{S}\left(w - \frac{k}{2}\right)$$

we have

$$\frac{df}{dS} = -\frac{3}{3}\sqrt{S}$$

so that

$$w = \frac{k'}{\sqrt{S}} - \frac{S}{2} + \frac{k}{2} \qquad \text{where } k' = \mu\left(\frac{\tilde{\beta}}{\alpha_0}\right)\left(4 + \frac{\mu^2\beta'}{2}\right)$$

Therefore for μ and ε small enough

$$S(0) = 2^{2/3}\mu^2 + O(\varepsilon^{1/3} + \mu^4)$$
$$T(0) = -2^{1/3}\mu^2 + O(\varepsilon^{1/3} + \mu^2)$$

In this way we obtain estimates for the perturbed trajectory Γ_ε

$$x = -\frac{1}{\alpha_0}(1-\gamma\varepsilon^{1/3}) \qquad \text{where} \quad \gamma = 2^{2/3} + O(\varepsilon^{1/3} + \mu^2)$$

$$z = \frac{1}{\alpha_0}(1-\gamma'\varepsilon^{1/3}) \qquad \text{where} \quad \gamma' = 2^{1/3} + O(\varepsilon^{1/3} + \mu^2)$$

which enable us to proceed further with the study of the perturbed stable manifold Γ_ε.

APPENDIX 3

The proof of the existence of the unstable manifold relies on the use of the Picard–Banach theorem in an appropriate functional space. To avoid problems due to the behavior of the flow far from the fixed point and to the existence of the invariant manifolds: $x^2+y^2=1$ and $z=0$, we consider a new flow which coincides with the previous one in a box containing the fixed point:

$$\frac{1}{\alpha(1+\varepsilon)^{1/2}} - \mu \leqslant x \leqslant \frac{1}{\alpha(1+\varepsilon)^{1/2}} + \mu\varepsilon$$

$$\frac{\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)} - \mu\varepsilon \leqslant y \leqslant \frac{\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)} + \mu\varepsilon$$

$$\frac{1}{3(1+\mu^2/36)^{1/2}} - \frac{1}{(1+\mu^2/4)^{1/2}} + \frac{(1+\varepsilon)^{1/2}}{\alpha} - \mu \leqslant z \leqslant \frac{(1+\varepsilon)^{1/2}}{\alpha} + \mu\varepsilon$$

That new flow is defined in such a way as to possess no invariant manifold and to have vanishing nonlinear terms far from the fixed point.

We build it as follows: P_ε has coordinates

$$\left[\frac{1}{\alpha(1+\varepsilon)^{1/2}}, \frac{\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}, \frac{(1+\varepsilon)^{1/2}}{\alpha}\right]$$

we introduce new variables:

$$X = x - \frac{1}{\alpha(1+\varepsilon)^{1/2}}$$

$$Y = y - \mu\frac{z}{2} - \frac{\mu\varepsilon(1+\varepsilon)^{1/2}}{2\alpha(2-\varepsilon)}$$

$$Z = z - \frac{(1+\varepsilon)^{1/2}}{\alpha}$$

The system is written then with inverted time $v = -u$:

$$\frac{dX}{dv} = -\frac{1}{\alpha(1+\varepsilon)^{1/2}}\left[\varepsilon + 2\left(1-\frac{1}{\alpha^2}\right)\right]X - \frac{2\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}\left(2-\frac{1}{\alpha^2}+\frac{\varepsilon}{2}\right)Y$$
$$-\frac{1}{\alpha(1+\varepsilon)^{3/2}}\left[\varepsilon + (\alpha^2-1)^{1/2}(2-\varepsilon)\left(1-\frac{1}{\alpha^2}+\frac{\varepsilon}{2}\right)\right]Z + \tilde{X}(X, Y, Z)$$
$$\frac{dY}{dv} = \frac{\mu\varepsilon(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}\left(1+\frac{1}{\alpha^2}\right)X + \frac{1}{\alpha(1+\varepsilon)^{1/2}}\left(2-\frac{\varepsilon}{\alpha^2}\right)Y$$
$$-\frac{\mu\varepsilon}{\alpha(2-\varepsilon)(1+\varepsilon)^{1/2}}\left(1+\frac{2-\varepsilon}{2\alpha^2}\right)Z + \tilde{Y}(X, Y, Z, \varepsilon, \mu)$$
$$\frac{dZ}{dv} = \frac{2(1+\varepsilon)^{1/2}}{\alpha^3}X + \frac{2\mu(1+\varepsilon)^{3/2}}{\alpha^3(2-\varepsilon)}Y + \frac{\mu^2(1+\varepsilon)^{3/2}}{\alpha^3(2-\varepsilon)}Z + \tilde{Z}(X, Y, Z, \varepsilon, \mu)$$

$\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ are nonlinear polynomial terms depending on the parameters μ, ε.

We define a new flow on $\mathscr{R}^3$ by

$$\frac{dX}{dv} = \text{linear terms} + C(X, Y, Z)\,\tilde{X}$$
$$\frac{dY}{dv} = \frac{1}{\alpha(1+\varepsilon)^{1/2}}\left(2-\frac{\varepsilon}{\alpha^2}\right)Y$$
$$+ C(X, Y, Z)\left[\frac{\mu\varepsilon(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}\left(1+\frac{1}{\alpha^2}\right)X - \frac{\mu\varepsilon}{\alpha(2-\varepsilon)(1+\varepsilon)^{1/2}}\right.$$
$$\left.\times\left(1+\frac{2-\varepsilon}{2\alpha^2}\right)Z + \tilde{Y}\right]$$
$$\frac{dZ}{dv} = \text{linear terms} + C(X, Y, Z)\,\tilde{Z}$$

where C is a cutoff function.

We shall now prove by using an appropriate function C that the fixed point has a C^1 stable (remember that time is inverted) manifold $Y(X, Z, \varepsilon, \mu)$ satisfying $\sup |Y| < \mu\varepsilon$.

We choose $C(X, Y, Z) = f(X)\,g(Y)\,h(Z)$ where:

(i) f is a positive C^∞ function with compact support $[-7\mu/\varepsilon', 7\mu\varepsilon/\varepsilon']$, equal to 1 on $[-\mu, \mu\varepsilon]$, smaller than 1 outside that interval, which satisfies for $\mu < \mu_0(\varepsilon')$ and $\varepsilon < \varepsilon_0(\mu)$: $-2\varepsilon' \leqslant X(df/dx) + f \leqslant 1$ where ε' is some given small real number.

(ii) g is a positive C^∞ function with compact support which is equal to 1 on $[-\varepsilon\mu, \varepsilon\mu]$ and smaller than 1 outside that interval.

(iii) h is a positive C^∞ function with compact support, equal to 1 on $[1/3(1+\mu^2/36)^{1/2} - 1/(1+\mu^2/4)^{1/2} - \mu, \mu\varepsilon]$ and smaller than 1 outside that interval, which satisfies $\varepsilon' > Zh > -\frac{2}{3}(1+\varepsilon')$ and $\frac{4}{9}(1+\varepsilon') \geqslant Z^2 h \geqslant 0$ for $\mu < \mu_0(\varepsilon')$ and $\varepsilon < \varepsilon_0(\mu)$.

Then the flow is defined on $\mathscr{R}^3$ and for $\mu_0(\varepsilon')$ small enough the invariant manifolds $x^2 + y^2 = 1$ and $z = 0$ no longer exist.

Let $Y(X, Z)$ be any C^0 bounded function defined on $\mathscr{R}^2$ and satisfying $\sup |Y| \leqslant \mu\varepsilon/4$, $Y(0, 0) = 0$.

We define $X(v; X_0, Z_0, Y)$ and $Z(v; X_0, Z_0, Y)$ as the solutions of

$$\frac{dX}{dv} = -\frac{1}{\alpha(1+\varepsilon)^{1/2}}\left[\varepsilon + 2\left(1 - \frac{1}{\alpha^2}\right)\right] X - \frac{2\mu(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}\left(2 - \frac{1}{\alpha^2} + \frac{\varepsilon}{2}\right) Y(X, Z)$$
$$-\frac{1}{\alpha(1+\varepsilon)^{3/2}}\left[\varepsilon + (\alpha^2 - 1)(2-\varepsilon)\left(1 - \frac{1}{\alpha^2} + \frac{\varepsilon}{2}\right)\right] Z$$
$$+ f(X)\, g(Y(X, Z))\, h(Z)\, \tilde{X}(X, Y(X, Z), Z, \varepsilon, \mu)$$

$$\frac{dZ}{dv} = \frac{2(1+\varepsilon)^{1/2}}{\alpha^3} X + \frac{2\mu(1+\varepsilon)^{3/2}}{\alpha^3(2-\varepsilon)} Y(X, Z) + \frac{\mu^2(1+\varepsilon)^{3/2}}{\alpha^3(2-\varepsilon)} Z$$
$$+ f(X)\, g(Y(X, Z))\, h(Z)\, \tilde{Z}(X, Y(X, Z), Z, \varepsilon, \mu)$$

with initial conditions $X(0) = X_0$, $Z(0) = Z_0$

$Y(X, Z)$ is a center stable manifold of $(0, 0, 0)$ if and only if it is a solution of

$$Y(X_0, Z_0) = \int_\infty^0 K[X(v; X_0, Z_0, Y), Y[X(v; X_0, Z_0, Y),$$
$$\times Z(v; X_0, Z_0, Y), Z(v; X_0, Z_0, Y)$$
$$* \exp\left[-\frac{2 - \varepsilon/\alpha^2}{\alpha(1+\varepsilon)^{1/2}} v\right] dv$$

$$K(X, Y, Z) = C(X, Y, Z)\left[\frac{\mu\varepsilon(1+\varepsilon)^{1/2}}{\alpha(2-\varepsilon)}\left(1 + \frac{1}{\alpha^2}\right) X\right.$$
$$\left. - \frac{\mu\varepsilon}{\alpha(2-\varepsilon)(1+\varepsilon)^{1/2}}\left(1 + \frac{2-\varepsilon}{2\alpha^2}\right) Z + \tilde{Y}(X, Y, Z)\right]$$

for any X_0, Z_0.

We must then apply the Picard–Banach theorem to the corresponding map T defined on the space of C^0 functions satisfying $Y(0,0)=0$ and $\sup|Y| \leqslant \mu\varepsilon/4$ which is a Banach space for the C^0 topology.

(i) One easily verifies that T maps this space into itself.
(ii) To prove that T is contracting one proceeds as follows: We set for any two functions Y and Y'

$$\delta X(X_0, Z_0, Y, Y') = X(X_0, Z_0, Y') - X(X_0, Z_0, Y)$$
$$\delta Z(X_0, Z_0, Y, Y') = Z(X_0, Z_0, Y') - Z(X_0, Z_0, Y)$$
$$\delta Y(X, Z) = Y'(X, Z) - Y(X, Z)$$

Owing to the choice of the cutoff function C, one has for $\mu_0(\varepsilon')$ small enough

$$\left|\frac{d\delta X}{dv}\right| \leqslant \tfrac{4}{3}(1+\varepsilon')\,|\delta E| + \varepsilon'\,|\delta Z| + \mu\,\|\delta Y\|_{C^0}$$

$$\left|\frac{d\delta Z}{dv}\right| \leqslant 2(1+2\varepsilon')\,|\delta X| + \varepsilon'\,|\delta Z| + \mu\,\|\delta Y\|_{C^0}$$

Therefore $M = [(\delta X)^2 + (\delta Z)^2]^{1/2}$ satisfies for ε' given small enough the differential inequality:

$$\frac{dM^2}{dv} \leqslant 2\sqrt{2}\,\mu\,\|\delta Y\|_{C^0}\,M + \frac{M^2}{3}\,[8+(13)^{1/2}]$$

with the condition $M(0)=0$

Let $N(v)$ be the C^1 function solution of

$$\frac{dN}{dv} = \tfrac{1}{6}[8+(13)^{1/2}]\,N + \sqrt{2}\,\mu\,\|\delta Y\|_{C^0}$$

with initial conditions $N(0) = k\mu\,\|\delta Y\|_{C_0}$ k any given positive real number

Then $N(v) = \mu\,\|\delta Y\|_{C^0}\left\{\left[k + \dfrac{6\sqrt{2}}{8+(13)^{1/2}}\right] e^{(v/6)[8+(13)^{1/2}]} - \dfrac{6\sqrt{2}}{8+(13)^{1/2}}\right\}$

and for all time $M(v) < N(v)$

Therefore

$$M(v) \leqslant \mu\,\|\delta Y\|_{C^0}\left[\frac{6\sqrt{2}}{8+(13)^{1/2}}\right] e^{(v/6)[8+(13)^{1/2}]}$$

a bound which does not depend on X_0 and Z_0.

Using that result one may prove after lengthy but straightforward calculations that

$$\|TY' - TY\|_{C^0} \leqslant \|Y' - Y\|_{C^0} \int_{\infty}^{0} \left[A\mu^2\varepsilon \exp\left\{\frac{v}{6}\,[8 + (13)^{1/2}]\right\} + B(\varepsilon + \mu)\right] \times \exp\left\{-\frac{1}{\alpha(1+\varepsilon)^{1/2}}\left(2 - \frac{\varepsilon}{\alpha^2}\right) v\right\} dv$$

with A and B some positive real numbers.

That inequality ensures contraction for $\mu_0(\varepsilon)$ and $\varepsilon(\mu)$ small enough.

We have thus found a unique C^0 invariant manifold, close to $y = \mu(z/2)$ since $\|Y\|_{C^0} \leqslant \varepsilon\mu/4$, which is the center stable manifold of the fixed point located at the origin for the modified system in variables X, Y, Z.

Moreover T maps the set of C^1 functions Y satisfying $Y(0, 0) = 0$, $\|Y\|_{C^0} \leqslant \mu\varepsilon/4$, $\|dY/dX\|_{C^0} \leqslant k'\mu\varepsilon$, $\|dY/dZ\|_{C^0} \leqslant k'\mu\varepsilon$ into itself for any positive real number k' large enough:

Therefore the fixed point lies in the C^0 closure of C^1 functions with derivatives bounded by $k'\mu\varepsilon$ and one may prove that it is not only uniformly Lipschitz but even C^1.[10,11]

That unique C^1 invariant manifold coincides locally with the unstable manifold of the origin. Therefore it is nothing but the unstable manifold of the origin for the modified system in variables X, Y, Z.

Reversing to the initial system (with usual time) we have thus proven that the fixed point P_ε has an unstable manifold $y = \mu(z/2) + \Psi_{\varepsilon,\mu}(x, z)$ defined on the rectangle:

$$\frac{1}{\alpha(1+\varepsilon)^{1/2}} - \mu \leqslant x \leqslant \frac{1}{\alpha(1+\varepsilon)^{1/2}} + \mu\varepsilon$$

$$\frac{1}{3(1+\mu^2/36)^{1/2}} - \frac{1}{(1+\mu^2/4)^{1/2}} + \frac{(1+\varepsilon)^{1/2}}{\alpha} - \mu \leqslant z \leqslant \frac{(1+\varepsilon)^{1/2}}{\alpha} + \mu\varepsilon$$

with $\Psi_{\varepsilon,\mu}$ a C^1 function satisfying

$$0 \leqslant \Psi_{\varepsilon,\mu}(x, z) \leqslant \frac{\mu\varepsilon}{2}$$

Actually one could even prove that owing to the polynomial nature of the vector field in those coordinates, $\Psi_{\varepsilon,\mu}$ is an analytic function.[11]

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